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The affine connectionof supersymmetric SO(N)/Sp(N) theories

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Abstract: We study the covariance properties of the equations satisfied by the generating functions of the chiral operators R and T of supersymmetric SO(N)/Sp(N) theories with symmetric/antisymmetric tensors. It turns out that T is an affine connection. As such it cannot be integrated along cycles on Riemann surfaces. This explains the discrepancies observed by Kraus and Shigemori. Furthermore, by means of the polynomial defining the Riemann surface, seen as quadratic-differential, one can construct an affine connection that added to T leads to a new generating function which can be consistently integrated. Remarkably, thanks to an identity, the original equations are equivalent to equations involving only one-differentials. This provides a geometrical explanation of the map recently derived by Cachazo in the case of Sp(N) with antisymmetric tensor. Finally, we suggest a relation between the Riemann surfaces with rational periods which arise in studying the laplacian on special Riemann surfaces and the integrality condition for the periods of T.

Keywords: Nonperturbative Effects, Supersymmetric Effective Theories.

In this note we consider the problem of the contributions to the free energy at order h, with h the dual Coxeter number, which arises in the case of supersymmetric $\mathrm{SO}(N)/\mathrm{Sp}(N)$ theories with symmetric/antisymmetric tensors [1, 2, 3]. Recently, in [4], Cachazo derived a map from $\mathrm{Sp}(N)$ to a $\mathrm{U}(N+2n)$ theory with one adjoint and a degree $n+1$ tree level superpotential. In his construction he observed interesting geometrical questions, such as the unusual phenomenon that vanishing of the period of T through a given cut does not imply that the cut closes on-shell. There are also other unusual properties.

Here we show that the chiral operator T of supersymmetric $\mathrm{SO}(N)/\mathrm{Sp}(N)$ theories with symmetric/antisymmetric tensors is an affine connection and so it cannot be consistently integrated along cycles on the Riemann surface. This is the geometrical origin of the discrepancies observed by Kraus and Shigemori. In our construction we will see that covariance forces us to consider the polynomial defining the Riemann surface as a quadratic-differential. By means of such a differential we construct an affine connection that added to T leads to a new generating function $\tilde{T}$ which is now a true one-differential and can be consistently integrated. We then show that, thanks to a remarkable identity, the original equations are equivalent to equations involving only one-differentials. We will conclude by suggesting a relation between the special Riemann surfaces with rational periods introduced in [5] and the integrality condition for the periods of T.

Let us start by considering the $\mathcal{N}=1$ supersymmetric theory with gauge group $\mathrm{Sp}(N)$ with a chiral superfield in the antisymmetric representation. The classical superpotential is

$$W_{\mathrm{tree}} = \sum_{j=0}^{n} \frac{g_j}{j+1} \operatorname{Tr} \Phi^{j+1} . \tag{1}$$

The generating functions of chiral operators

$$T(z) = \operatorname{Tr} \frac{1}{z-\Phi} , \qquad R(z) = -\frac{1}{32\pi^2} \operatorname{Tr} \frac{W_\alpha W^\alpha}{z-\Phi} , \tag{2}$$

satisfy the equations

$$[W'R]_- = \frac{1}{2} R^2 , \qquad [W'T]_- = TR + 2R' . \tag{3}$$

Usually, both T and R are integrated over cycles on the Riemann surface. It follows that both R and T should be one-differentials. Let us show that this is not the case. More precisely, we show that the unique consistent geometric interpretation of (3) is that R be a one-differential and T an affine connection. Actually, note that the unique interpretation consistent with the first equation in (3) is that R be a one-differential. Let us now consider the right hand side of the second equation in (3). This contains the first derivative of the one-differential R. In the intersection between two patches we have

$$R_\alpha(z_\alpha) = \frac{dz_\beta}{dz_\alpha} R_\beta(z_\beta) , \tag{4}$$

so that the transformation of R' is

$$\frac{d}{dz_\alpha} R_\alpha(z_\alpha) = \left(\frac{dz_\beta}{dz_\alpha}\right)^2 \frac{d}{dz_\beta} R_\beta(z_\beta) + \frac{d^2 z_\beta}{dz_\alpha^2} R_\beta(z_\beta) . \tag{5}$$

In order that $[W'T]_- = TR + 2R'$ be covariantly defined, we should relax the condition that T be a one-differential. Actually, we should require that $TR + 2R'$ be a quadratic differential, that is

$$T_\alpha(z_\alpha)R_\alpha(z_\alpha) + 2\frac{d}{dz_\alpha}R_\alpha(z_\alpha) = \left(\frac{dz_\beta}{dz_\alpha}\right)^2 \left[T_\beta(z_\beta)R_\beta(z_\beta) + 2\frac{d}{dz_\beta}R_\beta(z_\beta)\right]. \tag{6}$$

Eqs. (5) and (6) give

$$T_\beta(z_\beta)\frac{dz_\beta}{dz_\alpha} = T_\alpha(z_\alpha) + 2\frac{dz_\alpha}{dz_\beta}\frac{d^2 z_\beta}{dz_\alpha^2}. \tag{7}$$

Recalling that an affine connection on a Riemann surface Σ is a set of functions $\mathcal{A}$, each one defined on a patch, such that in the non-empty intersection $U_\alpha \cap U_\beta$

$$\mathcal{A}_\beta(z_\beta)\frac{dz_\beta}{dz_\alpha} = \mathcal{A}_\alpha(z_\alpha) + \frac{d}{dz_\alpha}\ln\frac{dz_\beta}{dz_\alpha}, \tag{8}$$

we see that T turns out to be twice an affine connection. As such it cannot be consistently integrated on cycles on the Riemann surface. This explains the discrepancies found between the gauge theory calculations and the ones obtained by the loop equation formulations. This also explains why one obtains the geometrical strange effect that the vanishing of the period of $T(z)dz$ is non zero but the cut closes up on-shell.

In order to write a covariantized version of eq. (3) we can use the procedure introduced in [6] in formulating the KdV equation on a Riemann surface, leading to W-algebras, and further developed in [7] to introduce higher order cocycles on Riemann surfaces. Furthermore, it has been shown in [7] that this covariantization is strictly related to uniformization and Liouville theories. Let us shortly review it. Let us consider the negative power of the Poincaré metric

$$e^{-k\varphi} = |J_H^{-1\prime}|^{-2k}\left(\frac{J_H^{-1} - \overline{J_H^{-1}}}{2i}\right)^{2k}, \tag{9}$$

where J_H^{-1} is the inverse of the uniformizing map from the upper half plane to Σ. One can easily check that the negative powers of the Poincaré metric satisfy the higher order generalization of nullvector equations, that is [7]

$$\mathcal{S}^{(2k+1)}_{J_H^{-1}} \cdot e^{-k\varphi} = 0, \qquad k = 0, \frac{1}{2}, 1, \dots, \tag{10}$$

with $\mathcal{S}^{(2k+1)}_h$ the higher order covariant Schwarzian operator

$$\mathcal{S}^{(2k+1)}_h = (2k+1)(h')^k\partial_z(h')^{-1}\partial_z(h')^{-1}\dots\partial_z(h')^{-1}\partial_z(h')^k, \tag{11}$$

where the number of derivatives is $2k+1$. The univalence of J_H^{-1} implies holomorphicity of the $\mathcal{S}^{(2k+1)}_{J_H^{-1}}$ operators. Eq. (10) is manifestly covariant and singlevalued on Σ. Furthermore it can be proved that the dependence of $\mathcal{S}^{(2k+1)}_h$ on h appears only through $\mathcal{S}^{(2)}_h \cdot 1 = \{h, z\}$ and its derivatives; for example

$$\mathcal{S}^{(3)}_{J_H^{-1}} = 3\left(\partial_z^3 + 2T^F\partial_z + T^{F\prime}\right), \tag{12}$$

which is the second symplectic structure of the KdV equation, where now T^F is the Fuchsian projective connection [7].

In the above derivation we used the polymorphic vector fields $1/J_H^{-1'}$ to construct the covariant operators $\mathcal{S}^{(2k+1)}_{J_H^{-1}}$ mapping $(-k,n)$-differentials, to $(k+1,n)$-differentials. In the present situation we should covariantize the quantity R'. However, note that we cannot use the inverse map of uniformization. The reason is that covariant operators constructed in terms of J_H^{-1} are singlevalued only in the case in which the operator is of order greater than 2. So we need a vector field which also encodes the geometrical information of the underlying Riemann surfaces. To solve this question we notice that $y^2 = W'^2 + f$ contains all the geometrical data. Also, note that covariance forces us to consider f as quadratic differential and so y^2 itself is a quadratic differential. It follows that the natural candidate to covariantize the R' is to use the vector field $\frac{1}{y}$. As in the covariantization reviewed above, the covariantized derivative should be homogeneous in y and map the one-differential R to a quadratic differential. This unequivocally leads to

$$\mathcal{D}_z \equiv y\frac{d}{dz}y^{-1} = \frac{d}{dz} - \frac{y'}{y}\,, \tag{13}$$

so that

$$\frac{d}{dz}R \longrightarrow \mathcal{D}_z R\,. \tag{14}$$

Let us now rewrite the right hand side of the second equation in (3) in the following form

$$TR + 2R' = (T + \delta T)R + 2\mathcal{D}_z R\,. \tag{15}$$

It follows that

$$\delta T = \frac{d}{dz}\ln y^2\,. \tag{16}$$

Now note that since both $TR + 2R'$ and $\mathcal{D}_z R$ are quadratic differentials, and R is a one-differential, it follows that

$$\tilde{T} = T + \frac{d}{dz}\ln y^2\,, \tag{17}$$

is a well-defined one-differential and can be consistently integrated along cycles on the Riemann surface.

Note that $y = W' - R$ is covariantly constant, that is the analogue of (10) is

$$\mathcal{D}_z(W' - R) = 0\,. \tag{18}$$

It follows that the second equation in (3) can be equivalently written in the form

$$[W'(\tilde{T} - \delta T)]_- = \tilde{T}R + 2\mathcal{D}_z W'\,. \tag{19}$$

We now show that

$$[W'\delta T]_- = -2\mathcal{D}_z W'\,, \tag{20}$$

so that the original equation $[W'T]_- = TR + 2R'$ is equivalent to

$$[W'\tilde{T}]_- = \tilde{T}R\,, \tag{21}$$

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which is covariant and involves only one-differentials. Let us give a closer look to eq. (20). It expresses the fact that the covariant derivative of W' is obtained by the negative frequencies of $W'\delta T$. It can be also written in the form

$$[W'\frac{y'}{y}]_- = W'\frac{y'}{y} - W'' \,, \tag{22}$$

that is

$$W'' = [W'\frac{y'}{y}]_+ \,. \tag{23}$$

On the other hand, differentiating $y^2 = W'^2 + f$, we have

$$W'\frac{y'}{y} = W'' + \frac{1}{2}\frac{W'f' - 2W''f}{W'^2 + f} \,. \tag{24}$$

The second term in the right hand side has the form P_{2n-2}/P_{2n}, with P_j denoting a polynomial of order j. Expanding in powers of $1/z$, we see that

$$\Big[\frac{W'f' - 2W''f}{W'^2 + f}\Big]_+ = 0 \,, \tag{25}$$

so that (23) is satisfied and the original equation is equivalent to its covariant form eq. (21).

Let us now consider the case of gauge group SO(N) with a symmetric tensor. In this case we have

$$[W'R]_- = \frac{1}{2}R^2, \qquad [W'T]_- = TR - 2R' \,. \tag{26}$$

Repeating the above derivation, we now see that consistency implies that in this case T transforms as -2 times an affine connection, that is

$$T_\beta(z_\beta)\frac{dz_\beta}{dz_\alpha} = T_\alpha(z_\alpha) - 2\frac{dz_\alpha}{dz_\beta}\frac{d^2 z_\beta}{dz_\alpha^2} \,. \tag{27}$$

It follows that the correct one-differential to integrate is

$$\tilde{T} = T - \frac{d}{dz}\ln y^2 \,, \tag{28}$$

whereas the eqs. (19) and (20) become

$$[W'(\tilde{T} - \delta T)]_- = \tilde{T}R - 2\mathcal{D}_z W' \,, \tag{29}$$

and

$$[W'\delta T]_- = 2\mathcal{D}_z W' \,, \tag{30}$$

which is satisfied because now $\delta T = -\frac{d}{dz}\ln y^2$. It follows that also in this case we have

$$[W'\tilde{T}]_- = \tilde{T}R \,. \tag{31}$$

We saw that while classically one has $f = 0$, so that $R = 0$ and T is a one-form on the Riemann sphere, quantum mechanically T has anomalous transformations. This can be modified to a one-form $\tilde{T}$ that can be integrated along cycles. However, since the original

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T cannot be integrated, one should understand whether the conditions $\oint_{A_i} T = N_i$ and $\oint_{B_i} T = b_i$ in [4] are essential. Actually, it seems that Cachazo's results can be obtained by imposing the weaker conditions[1]

$$\mathrm{Tr}\, \mathbf{1} = \mathrm{Tr}\, \mathbf{1}_U - 2n\,, \tag{32}$$

$$\frac{1}{2\pi i} \oint_{A_i} T_U = N_i + 2\,, \qquad i = 1, \ldots, n\,, \tag{33}$$

that do not imply T to be a one-form. So, essentially, the results in [4] do not use the integrality condition on the cycles of the original T. This is different with respect to the previous investigations based on the loop equation [2]. Understanding how the results in [4] can be obtained in the original formulation of Kraus and Shigemori [1] remains an interesting open question. Even if a detailed and clarifying investigation of the ambiguities has been given in [3], it seems that there are still interesting aspects to be fully understood. In this respect it is interesting to observe that besides the appearance of the affine connection, $\mathrm{SO}(N)/\,\mathrm{Sp}(N)$ theories have another distinguished, perhaps related, feature concerning the algebraic relations satisfied by S. In the case of $\mathrm{SU}(N)$ one has [8]

$$S^N = \Lambda^{3N} + \{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}\,, \tag{34}$$

In [1] it was suggested that such kind of relations may be at the origin of the observed discrepancy. However in [8] it was pointed out that the identity (34) of the microscopic $\mathrm{SU}(N)$ theory corresponds to the field equation $S^N = \Lambda^{3N}$ one obtains as stationary points of the superpotential. This dual description is similar to the magnetic description of free electrodynamics where the Bianchi identity arises only on-shell. In this respect, as observed by Cachazo [4], if this picture holds, since the effect of the field equation is not directly visible in the off-shell low energy effective superpotential, it follows that the identity should not be at the origin of the discrepancy observed in [1]. However, there is another identity besides eq. (34), that is [9]

$$(S^N - \Lambda^{3N})^N = 0\,. \tag{35}$$

It is interesting to find this relation in the case of a generic simple group G. First recall that since S is a bilinear in fermionic fields, classically we have [8]

$$S^{\dim(G)+1} = 0\,. \tag{36}$$

On the other hand [8, 10]

$$S^h = \{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}\,, \tag{37}$$

where h is the dual Coxeter number of G. This relation gets instantonic corrections [8, 10]

$$S^h = c(G)\Lambda^{3h} + \{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}\,. \tag{38}$$

Now note that the $X^{\dot\alpha}$ in (37) and (38) can differ only by a chiral operator: dimensional analysis and R-symmetry forbid the existence of terms $\{\overline{Q}_{\dot\alpha}, \delta X^{\dot\alpha}\}$ that vanish as $\Lambda \to 0$.

[1]We follow the notation in [4].

However, while how observed in [9] in the case of $\mathrm{SU}(N)$ the last two identities imply $\{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}^N = 0$, that is eq. (35), for other groups there is an interesting new phenomenon. Namely, for general groups we have

$$\{\overline{Q}_{\dot\alpha}, X^{\dot\alpha}\}^M = 0\,, \tag{39}$$

where M is the lowest integer such that $hM \geq \dim(G) + 1$. Eq. (39) implies

$$(S^h - c(G)\Lambda^{3h})^M = 0\,, \tag{40}$$

that in the classical limit reduces to

$$S^{hM} = 0\,. \tag{41}$$

Of course, since $S^{\dim(G)+1} = 0$ implies $S^{hM} = 0$ even in the case in which $hM > \dim(G) + 1$, it follows that eqs. (36) and (41) are consistent. Nevertheless, it is interesting to observe that in general, as in the case of $\mathrm{Sp}(N)$ where $h = N/2 + 1$, one has[2] $hM > \dim(G) + 1$. Therefore, if $[\dim(G) + 1]/h$ is not an integer, then the quantum and classical exponents hM and $\dim(G) + 1$ do not coincide. It would be interesting to understand how this information is encoded in the dual description. Apparently, while on-shell we get the analogue of (38), there is no information about the relation (40). In particular, in the case of $\mathrm{Sp}(N)$ the matrix model effective theory does not seem to recognize the difference between the two critical exponents. We also note that whereas in chiral correlators eqs. (38) and (40) may lead to the same result, this is not the case of correlators which also contain non-chiral fields: according to (40) these would be identically vanishing.

Let us conclude this note by observing that the above geometrical structure may help in understanding the nature of the integrality condition on the periods of T. In this respect, we note that integrality condition emerged in studying the eigenfunctions of the laplacian on a Riemann surface. In particular, in [5] it has been shown that eigenvalues with a nontrivial dependence on the complex structure can be obtained as solutions of the equation

$$\omega_{n',m'} = c\,\omega_{n,m}\,. \tag{42}$$

This equation is equivalent to

$$m'_j - \sum_{k=1}^{h} \Omega_{jk} n'_k = \overline{c}\left(m_j - \sum_{k=1}^{h} \Omega_{jk} n_k\right), \qquad j = 1, \ldots, h\,, \tag{43}$$

where m_j, n_j, m'_j, n'_j are integers and Ω_{jk} is the Riemann period matrix. In [5] it has been derived a set of solutions of such an equation. The general problem involved in eq. (43) concerns the properties of the Riemann period matrix and its number theoretical structure leading to periods with rational entries. It can bee shown that such surfaces correspond to

[2] Since $N(N+1)/2 + 1 = h(N+3) + 4$, we see that only for $\mathrm{Sp}(10)$ one has $hM = \dim(G) + 1$. This special case is the first of the ones not explicitly investigated in [1] and [4]. It is interesting to observe that since for $h(\mathrm{SO}(N)) = N - 2$, so that $N(N+1)/2 + 1 = h(N+3)/2 + 4$, we have that also in this case $N = 10$ is the unique solution of $hM = \dim(G) + 1$.

branched covering of the torus. It would be interesting whether this is also the case of the Riemann surfaces having integer period for T. In this respect we note that holomorphic affine connections exist only on the torus. In the case of branched covering of the torus this connection has poles but if this originated from the holomorphic one of the base torus, this should reflect in peculiar properties related to the integrality condition.

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